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Stick pendulum via Lagrangian mechanics

by Frank Owen, 23 May 2014

Note: The following development follows from other developments shown for pendula via Lagrange mechanics to be found at www.aoengr.com/Dynamics.

Unlike the simple pendulum, which consists of a massless link holding a point mass without any resistance to rotation, a stick pendulum is a link of length ℓ with mass *m* and a rotational inertia about its mass center of

$$J = \frac{1}{12} \cdot m \cdot l^2$$

Figure 1 shows a stick pendulum.

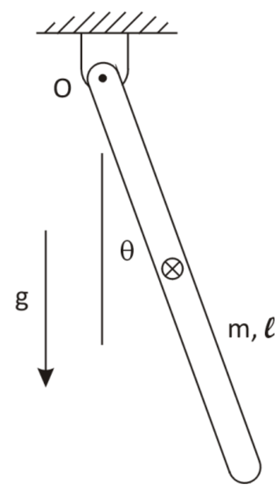

Figure 1 – Stick pendulum

Lagrangian formulation

For the Lagrangian formulation we need expressions for the kinetic energy *T* and the potential energy *U* of the pendulum. The kinetic energy consists of the movement of the pendulum's mass center and also the rotational energy of the stick itself.

$$T = \frac{1}{2} \cdot m \cdot v^2 + \frac{1}{2} \cdot J \cdot \dot{\theta}^2$$

The mass center moves with a speed

$$v = \frac{l}{2} \cdot \dot{\theta}$$

Thus

$$T = \frac{1}{8} \cdot m \cdot l^2 \cdot \dot{\theta}^2 + \frac{1}{24} \cdot m \cdot l^2 \cdot \dot{\theta}^2 = \frac{1}{6} \cdot m \cdot l^2 \cdot \dot{\theta}^2$$

The potential energy is purely gravitational and is similar to that of the simple pendulum except that the mass center only rises half as much as it does for a mass mounted at the pendulum end. Thus it is half that of a simple pendulum.

$$U = m \cdot g \cdot y = m \cdot g \cdot \frac{l}{2} \cdot (1 - \cos\theta)$$

where *y* is the distance of the mass center above its lowest position at $\theta = 0$. Thus the Lagrangian is

$$L = T - U = \frac{1}{6} \cdot m \cdot l^2 \cdot \dot{\theta}^2 - m \cdot g \cdot \frac{l}{2} \cdot (1 - \cos\theta)$$

Like the simple pendulum there is just one equation of motion, where $q_1 = \theta$. The Lagrange formulation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Now let's get the parts and pieces from the Lagrangian.

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3} \cdot m \cdot l^2 \cdot \dot{\theta}$$

Taking the time derivative of this

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{3} \cdot m \cdot l^2 \cdot \ddot{\theta}$$

Then

$$\frac{\partial L}{\partial \theta} = -m \cdot g \cdot \frac{l}{2} \cdot \sin\theta$$

Putting these parts together, we get the equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{1}{3} \cdot m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot \frac{l}{2} \cdot \sin\theta = 0$$

$$\frac{1}{3} \cdot l \cdot \ddot{\theta} + g \cdot \frac{1}{2} \cdot \sin\theta = 0$$

Or

$$\ddot{\theta} + \frac{3}{2} \cdot \frac{g}{l} \cdot \sin\theta = 0$$