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Pendulum on a cart via Lagrangian mechanics

by Frank Owen, 23 May 2014

Note: The development here follows on the previous development of the equation of motion of a simple pendulum via Lagrangian mechanics. See www.aoengr.com/Dynamics for that explanation.

Now we develop further the problem of equations of motion of a pendulum by placing the simple pendulum on a cart travelling on a frictionless track (Figure 1). The cart is driven back and forth on the track by an external force acting along the path of the track. This changes the Lagrangian formulation by 1) introducing another degree of freedom to the problem and 2) by adding a non-conservative force to the problem.

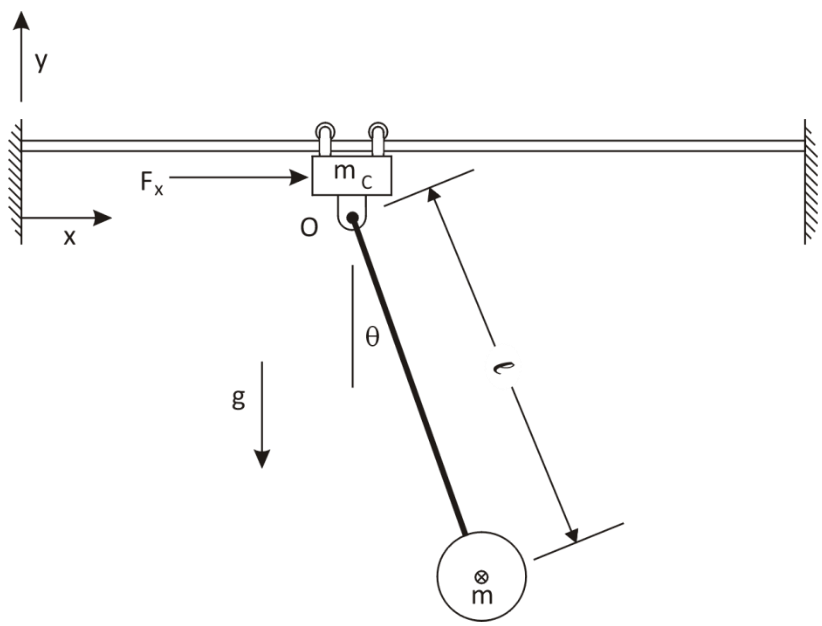

Figure 1 – Pendulum on cart

Thus the pendulum is starting to look like an overhead or gantry crane.

Lagrangian formulation

For the total kinetic energy of the system, we shall need the speeds of the cart and of the pendulum bob. The cart velocity is

$$v_C = \dot{x}_C$$

so x_C is naturally our second generalized coordinate. The velocity of the pendulum mass $\vec{v}$ is a vector quantity consisting of the velocity due to the pendulum swinging about its pivot with the cart velocity superimposed upon it. Figure 2 shows a breakdown of the x and y components of the pendulum bob. We need just the magnitude of this velocity to get the kinetic energy of the pendulum bob.

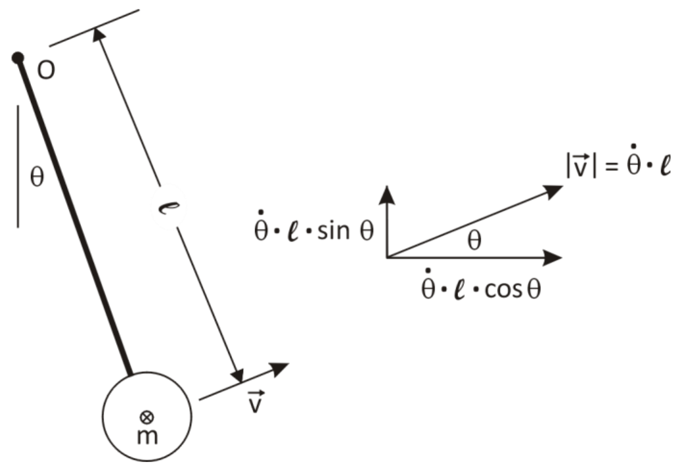

Figure 2 – Velocity of pendulum bob

$$\vec{v} = \vec{v}_\theta + \vec{v}_C = \left(\dot{x}_C + \dot{\theta} \cdot l \cdot \cos\theta\right) \cdot \hat{\imath} + \left(\dot{\theta} \cdot l \cdot \sin\theta\right) \cdot \hat{\jmath}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are, respectively, the x and y unit vectors. So

$$v^2 = \left(\dot{x}_C + \dot{\theta} \cdot l \cdot \cos\theta\right)^2 + \left(\dot{\theta} \cdot l \cdot \sin\theta\right)^2$$

$$v^2 = \dot{x}_C^2 + 2 \cdot \dot{x}_C \cdot \dot{\theta} \cdot l \cdot \cos\theta + \dot{\theta}^2 \cdot l^2 \cdot (\cos\theta)^2 + \dot{\theta}^2 \cdot l^2 \cdot (\sin\theta)^2$$

Simplifying

$$v^2 = \dot{x}_C^2 + 2 \cdot \dot{x}_C \cdot \dot{\theta} \cdot l \cdot \cos\theta + \dot{\theta}^2 \cdot l^2$$

From this, we can calculate T, the system's total kinetic energy.

$$T = T_C + T_B = \frac{1}{2} \cdot m_C \cdot v_C^2 + \frac{1}{2} \cdot m \cdot v^2$$

where B signifies the pendulum bob. Thus

$$T = \frac{1}{2} \cdot (m_C \cdot v_C^2 + m \cdot v^2) = \frac{1}{2} \cdot \left[m_C \cdot \dot{x}_C^2 + m \cdot \left(\dot{x}_C^2 + 2 \cdot \dot{x}_C \cdot \dot{\theta} \cdot l \cdot \cos\theta + \dot{\theta}^2 \cdot l^2\right)\right]$$

The potential energy is, again, just what it was before in the case of the simple pendulum. It is simply based on the height of the pendulum bob above the reference height, which we select as the bottom of the pendulum stroke. Thus

$$U = m \cdot g \cdot l \cdot (1 - \cos\theta)$$

So the Lagrangian is

$$L = T - U = \frac{1}{2} \cdot \left[m_C \cdot \dot{x}_C^2 + m \cdot \left(\dot{x}_C^2 + 2 \cdot \dot{x}_C \cdot \dot{\theta} \cdot l \cdot \cos\theta + \dot{\theta}^2 \cdot l^2\right)\right] - m \cdot g \cdot l \cdot (1 - \cos\theta)$$

This expression is much more complicated than that of the simple pendulum without the cart. We shall have to be careful with its differentiation, following the product and chain rules carefully, to develop the equations of motion correctly. The two generalized coordinates are x_C and θ. This leads to Lagrangian equations, which is to be expected with a 2DOF system. These equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_C}\right) - \frac{\partial L}{\partial x_C} = F_x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Now we must calculate the pieces. From the first equation

$$\frac{\partial L}{\partial \dot{x}_C} = m_C \cdot \dot{x}_C + m \cdot \left(\dot{x}_C + \dot{\theta} \cdot l \cdot \cos\theta\right)$$

Taking the derivative of this with respect to *t*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_C}\right) = m_C \cdot \ddot{x}_C + m \cdot \left(\ddot{x}_C + \ddot{\theta} \cdot l \cdot \cos\theta - \dot{\theta}^2 \cdot l \cdot \sin\theta\right)$$

Since x_C does not appear in the expression for *L*, the second term of the first equation disappears.

$$\frac{\partial L}{\partial x_C} = 0$$

Thus the first equation becomes

$$m_C \cdot \ddot{x}_C + m \cdot \left(\ddot{x}_C + \ddot{\theta} \cdot l \cdot \cos\theta - \dot{\theta}^2 \cdot l \cdot \sin\theta\right) = F_x$$

Or

$$(m_C + m) \cdot \ddot{x}_C + m \cdot l \cdot \left(\ddot{\theta} \cdot \cos\theta - \dot{\theta}^2 \cdot \sin\theta\right) = F_x$$

For the second equation

$$\frac{\partial L}{\partial \dot{\theta}} = m \cdot \left(\dot{x}_C \cdot l \cdot \cos\theta + \dot{\theta} \cdot l^2\right)$$

Taking the derivative of this with respect to time

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m \cdot \left(\ddot{x}_C \cdot l \cdot \cos\theta - \dot{x}_C \cdot l \cdot \sin\theta \cdot \dot{\theta} + \ddot{\theta} \cdot l^2\right)$$

Now for the second term in the second Lagrangian equation

$$\frac{\partial L}{\partial \theta} = -m \cdot \left(\dot{x}_C \cdot \dot{\theta} \cdot l \cdot \sin\theta\right) - m \cdot g \cdot l \cdot \sin\theta = -m \cdot \left(\dot{x}_C \cdot \dot{\theta} + g\right) \cdot l \cdot \sin\theta$$

Thus the second Lagrangian equation becomes

$$m \cdot \left(\ddot{x}_C \cdot l \cdot \cos\theta - \dot{x}_C \cdot l \cdot \sin\theta \cdot \dot{\theta} + \ddot{\theta} \cdot l^2\right) + m \cdot \left(\dot{x}_C \cdot \dot{\theta} + g\right) \cdot l \cdot \sin\theta = 0$$

Simplifying

$$\ddot{x}_C \cdot l \cdot \cos\theta + \ddot{\theta} \cdot l^2 + g \cdot l \cdot \sin\theta = 0$$

The two equations then are

$$(m_C + m) \cdot \ddot{x}_C + m \cdot l \cdot \left(\ddot{\theta} \cdot \cos\theta - \dot{\theta}^2 \cdot \sin\theta\right) = F_x$$

$$\ddot{x}_C \cdot l \cdot \cos\theta + \ddot{\theta} \cdot l^2 + g \cdot l \cdot \sin\theta = 0$$

Thus we have two second order ODEs in the two independent variables x_C and θ. Terms for each variable appear in the equation of the other variable, so the system is coupled.