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Simple pendulum via Lagrangian mechanics

by Frank Owen, 22 May 2014

The equation of motion for a simple pendulum of length l, operating in a gravitational field is

$$\ddot{\theta} + \frac{g}{l} \cdot sin\theta = 0$$

This equation can be obtained by applying Newton's Second Law (N2L) to the pendulum and then writing the equilibrium equation. It is instructive to work out this equation of motion also using Lagrangian mechanics to see how the procedure is applied and that the result obtained is the same.

For this example we are using the simplest of pendula, i.e. one with a massless, inertialess link and an inertialess pendulum bob at its end, as shown in Figure 1.

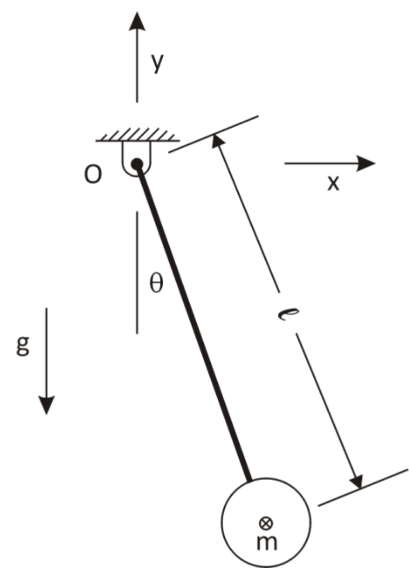

Figure 1 – Simple pendulum

Lagrangian formulation

The *Lagrangian function* is defined as

$$L = T - U$$

where *T* is the total kinetic energy and *U* is the total potential energy of a mechanical system.

To get the equations of motion, we use the Lagrangian formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

where *q* signifies generalized coordinates and *F* signifies non-conservative forces acting on the mechanical system. For the simplify pendulum, we assume no friction, so no non-conservative forces, so all F_i are 0. The aforementioned equation of motion is in terms of θ as a coordinate, not in terms of *x* and *y*. So we need to use kinematics to get our energy terms in terms of θ.

For *T*, we need the velocity of the mass.

$$v = l \cdot \dot{\theta}$$

So

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l \cdot \dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

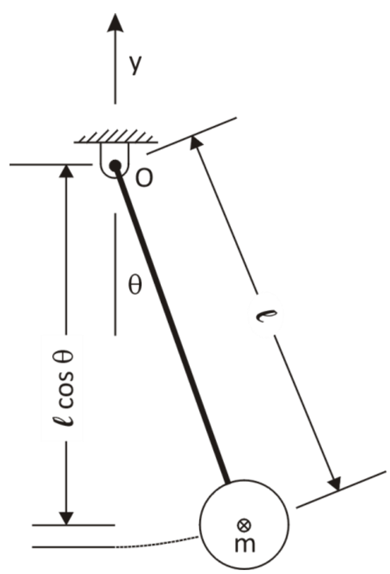

Figure 2 – Height for potential energy

The potential energy, *U*, depends only on the *y*-coordinate. Taking θ = 0 as the position where *U* = 0,

$$y = l - l \cdot \cos\theta = l \cdot (1 - \cos\theta)$$

Thus

$$U = m \cdot g \cdot y = m \cdot g \cdot l \cdot (1 - \cos\theta)$$

Now we have all the parts and pieces to complete the Lagrangian formulation. The Lagrangian function in terms of θ is

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - m \cdot g \cdot l \cdot (1 - \cos\theta)$$

The only generalized coordinate is $q_1 = \theta$. So

$$\frac{\partial L}{\partial \dot{\theta}} = m \cdot l^2 \cdot \dot{\theta}$$

Continuing

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m \cdot l^2 \cdot \ddot{\theta}$$

Then

$$\frac{\partial L}{\partial \theta} = -m \cdot g \cdot l \cdot \sin\theta$$

Now, putting these last two equations together

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta = 0$$

Simplifying,

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0$$

QED.

It is natural to ask why anyone would want to approach this using the Lagrangian formulation rather than just applying N2L, since that is rather straightforward. The answer nowadays is that the Lagrangian formulation, though tedious for humans, is methodical and lends itself well to automation with computers. Of course Lagrange came up with this long before the day of computers. He was French, and the French have a different way of thinking about just about everything.

Note: Further development of pendula of various configurations via Lagrangian mechanics is to be found at www.aoengr.com/Dynamics.