Alpha Omega Engineering, Inc.
872 Toro Street, San Luis Obispo, CA 93401 – (805) 541 8608 (home office), (805) 441 3995 (cell)

Momentum and velocity diagrams in collision analysis

by Frank Owen, Alpha Omega Engineering, Inc., 3 February 2013,
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Introduction

The momentum diagram is a graphical representation of the conservation of linear momentum principle. In a collision between two vehicles, linear momentum, $\vec{p}$, is preserved. So $\vec{p}_i = \vec{p}_f$. Thus

$$m_A \cdot \vec{V}_{Ai} + m_B \cdot \vec{V}_{Bi} = m_A \cdot \vec{V}_{Af} + m_B \cdot \vec{V}_{Bf}$$

Focusing on a single car, the final linear momentum is just the initial linear momentum with the impulse, $\hat{F}$, applied. So for each vehicle,

$$\vec{p}_i + \hat{F} = \vec{p}_f$$

The momentum diagram represents these two equations graphically. It allows the accident reconstructionist to use the initial and final linear momenta to determine the direction and magnitude of the impulses acting on each vehicle.

Calculation of pre-collision momentum from post-collision momentum

To illustrate the construction of a momentum diagram, we must remember the order in which the vectors are calculated:

1) The output momenta are calculated by applying conservation of energy along the spin-out trajectories to calculate the vehicle velocities just after impact. So $\vec{V}_{Af}$ and $\vec{V}_{Bf}$ are known, from which $\vec{p}_{Af}$ and $\vec{p}_{Bf}$ are known.
2) The conservation of linear momentum equation is then applied. From the final velocities (magnitude and direction) the initial velocities are to be found. The problem is that the unknown initial velocities contain four unknowns—that is two magnitudes and two directions—and the 2-D conservation-of-momentum equation only allows one to solve for two unknowns. Thus some other way must be found to solve for two of the four unknown quantities. If the initial direction of motion is known (from pre-impact skid marks or from their absence and the initial direction of travel of the accident vehicles), then there are only the two unknown magnitudes to find. This accounting of knowns and remaining unknowns will become clear with the construction of the momentum diagram.

These four linear momentum vectors can then be used to construct the impulse diagram.

An example serves to illustrate the process. Take the case illustrated in Figure 1.

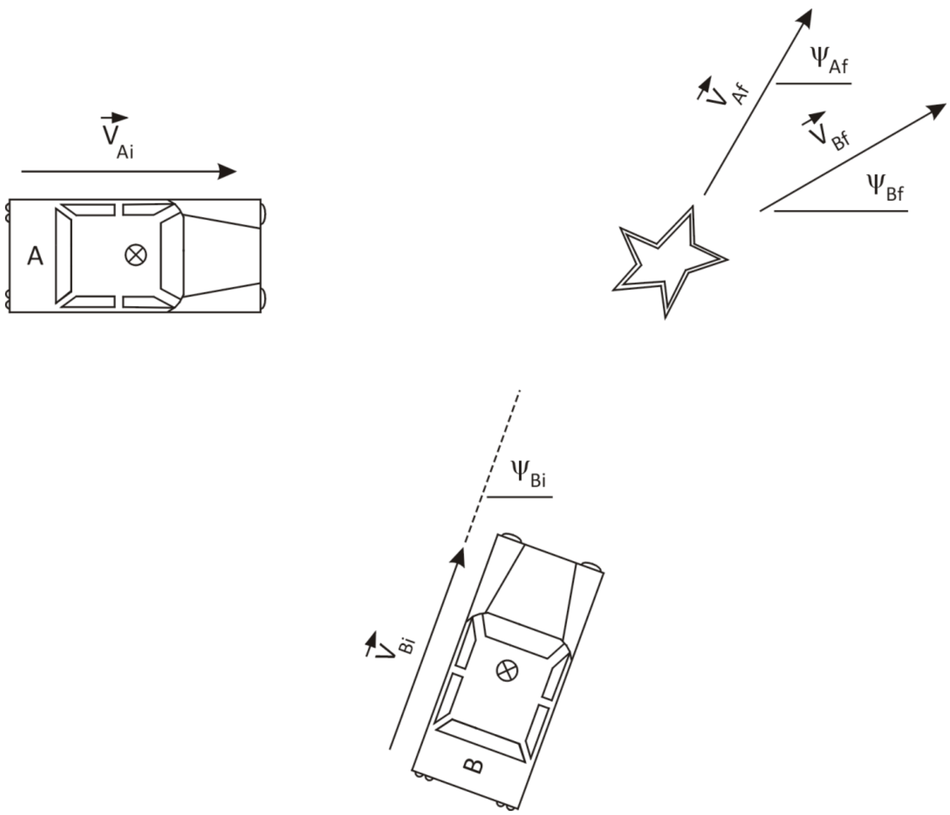


Figure 1 – Collision between two vehicles

From the post-collision spin-out evidence and an energy balance, one calculates $\vec{V}_{Af}$ and $\vec{V}_{Bf}$, and thus $\vec{p}_{Af}$ and $\vec{p}_{Bf}$. We start the momentum diagram with these vectors, as shown in Figure 2. The post-collision momentum is the sum of these two vectors, so their vector sum is graphically constructed as shown in the figure.

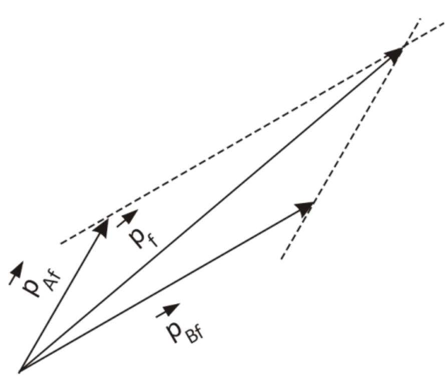


Figure 2 – Post-collision momentum vectors

Since the pre-crash momentum is equal to the post-crash momentum, $\vec{p}_f$, the $\vec{p}_i$ vector can be drawn as shown in Figure 3. $\vec{p}_i$ is the sum of $\vec{p}_{Ai}$ and $\vec{p}_{Bi}$. So now the possibilities for these two vectors become clear. If their directions are known, then one can construct the direction lines and fix their lengths so that their sum is $\vec{p}_i$, as shown below.

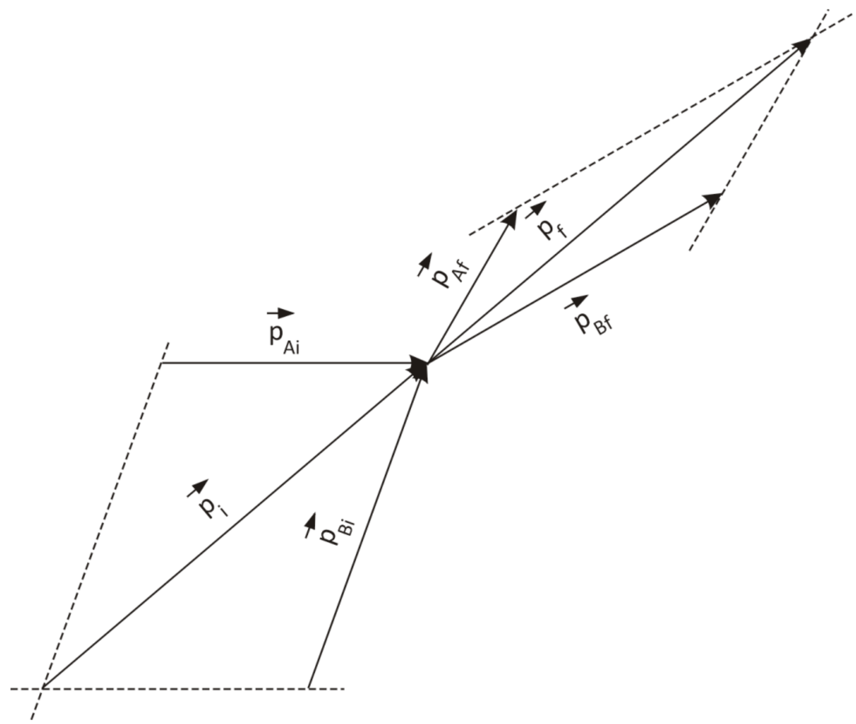


Figure 3 – Construction of pre-collision vectors from post-collision vectors

If, instead, one of the initial velocity vectors were known, one would draw it onto the figure and draw the second vector so that the sum of the two pre-collision vectors was $\vec{p}_i$. Another possibility is that the two pre-crash velocity magnitudes are known, perhaps from on-board data recorders. Unknown are the directions of the two pre-crash velocities. Thus the lengths of the two pre-collision momenta vectors would be known, and they would have to be arranged on the diagram so that their sum was again $\vec{p}_i$.

Calculation of impulse from momentum diagram

Once the initial and final momenta vectors are drawn, we can change our point of view and regard the second equation given in the introduction. This equation focuses on the change in momentum of each of the vehicles due to the impulse experienced during the collision. So

$$\vec{p}_{Ai} + \hat{F}_A = \vec{p}_{Af} \quad \text{and} \quad \vec{p}_{Bi} + \hat{F}_B = \vec{p}_{Bf}$$

The $\vec{p}$ vectors are already shown on the diagram in Figure 3. $\hat{F}_A$ is the impulse inflicted by vehicle B on vehicle A, and $\hat{F}_B$ is the impulse inflicted by vehicle A on vehicle B. By Newton's third law, $\hat{F}_A = -\hat{F}_B$. Thus we can construct the impulse vectors from the momentum vectors, as shown in Figure 4.

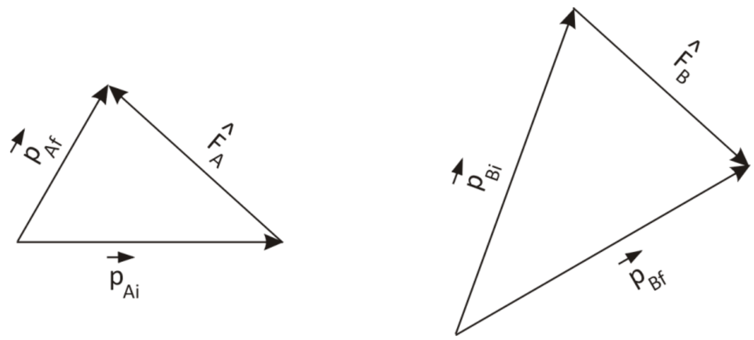


Figure 4 – Graphical construction of impulse vectors from momentum vectors

Note that the two impulse vectors are equal and opposite. This then gives the size of the impulse, its magnitude as well as its direction. The directions then give the principal direction of force between the two vehicles. One can then draw the impulse vectors on the original momentum diagram to complete it, as shown in Figure 5. The numbers show the order in which the vectors are drawn on the diagram.

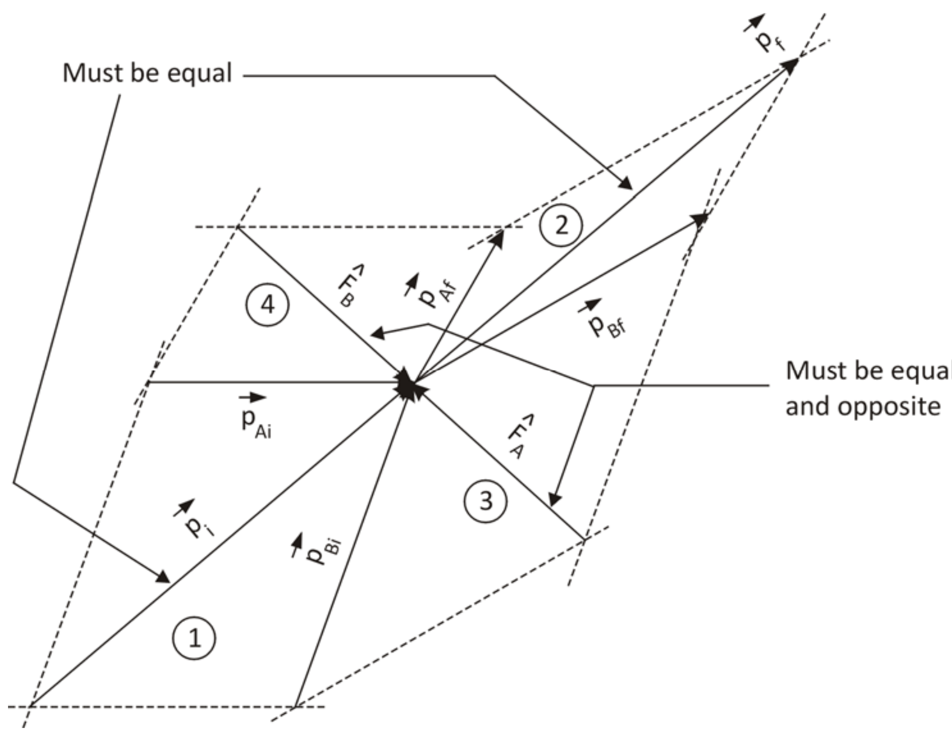
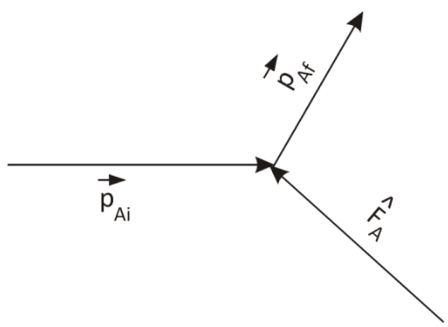


Figure 5 – Complete momentum diagram

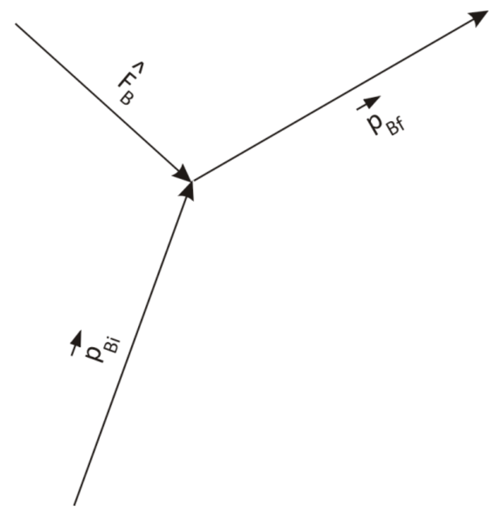
Figure 5 also shows the constraints on the momentum vector-pair and the impulse force-pair in the diagram. The four parallelograms shown follow from the vector sums contained in the momentum and impulse/momentum equations.

Now several other useful results can be obtained from the momentum diagram. The impulse vectors $\hat{F}_A$ and $\hat{F}_B$ give the so-called principle direction of force (PDOF). It is useful also to look at what is happening to each vehicle separately. Figure 6 is a view that is extracted from the momentum diagram.

It shows the effect of each vehicle on the other. It is a bit confusing because of the naming convention of the impulses. $\hat{F}_A$ is the impulse on A; that is, it is the impulse of B acting on A. Likewise $\hat{F}_B$ is the impulse acting on B, which, of course, comes from vehicle A.



(a) Effect of vehicle B on vehicle A



(b) Effect of vehicle A on vehicle B

Figure 6 – Effect of each vehicle on the other

This figure is useful as a check of the reasonableness of the solution. Prior to the collision, vehicle A is proceeding to the right. During the collision it is deflected upward to the right by the impulse, which is directed upward and to the left. Likewise, vehicle B is traveling prior to the collision upward to the right. After the collision it is traveling also upward to the right, though its upward momentum has been diminished by the downward component of $\hat{F}_B$, and its rightward component has been augmented by the rightward component of $\hat{F}_B$.

Another useful result that can be extracted from the momentum diagram is ΔV. The magnitude of the change in velocity is a measure of the severity of a crash. Delta-V is actually a vector quantity:

$$\Delta\vec{V} = \vec{V}_f - \vec{V}_i$$

We have already a related quantity. Recall that

$$\vec{p} = m \cdot \vec{V}$$

The relationship

$$\vec{p}_i + \hat{F} = \vec{p}_f$$

was used to construct Figure 4 above. Manipulating it,

$$\hat{F} = \vec{p}_f - \vec{p}_i = m\left(\vec{V}_f - \vec{V}_i\right)$$

So

$$\vec{V}_f - \vec{V}_i = \Delta\vec{V} = \frac{\hat{F}}{m}$$

Thus the impulse can simply be divided through by the vehicle's mass to get the vehicle's $\Delta\vec{V}$. If the analyst is interested only in ΔV, i.e. only the magnitude, then the magnitude of $\hat{F}$ can be used.

The crash's *coefficient of* restitution, e, can also be extracted from the momentum diagram. *e* is defined as

$$e = \frac{departure\ velocity}{approach\ velocity}$$

Figure 7 is a velocity diagram that shows graphically the meaning of *e*. This figure has also been extracted from the momentum diagram. It shows the pre- and post-crash velocities of each vehicle, which can be gotten simply by dividing the momentum of each vehicle, $\vec{p}$, by its *m*. Care must be taken

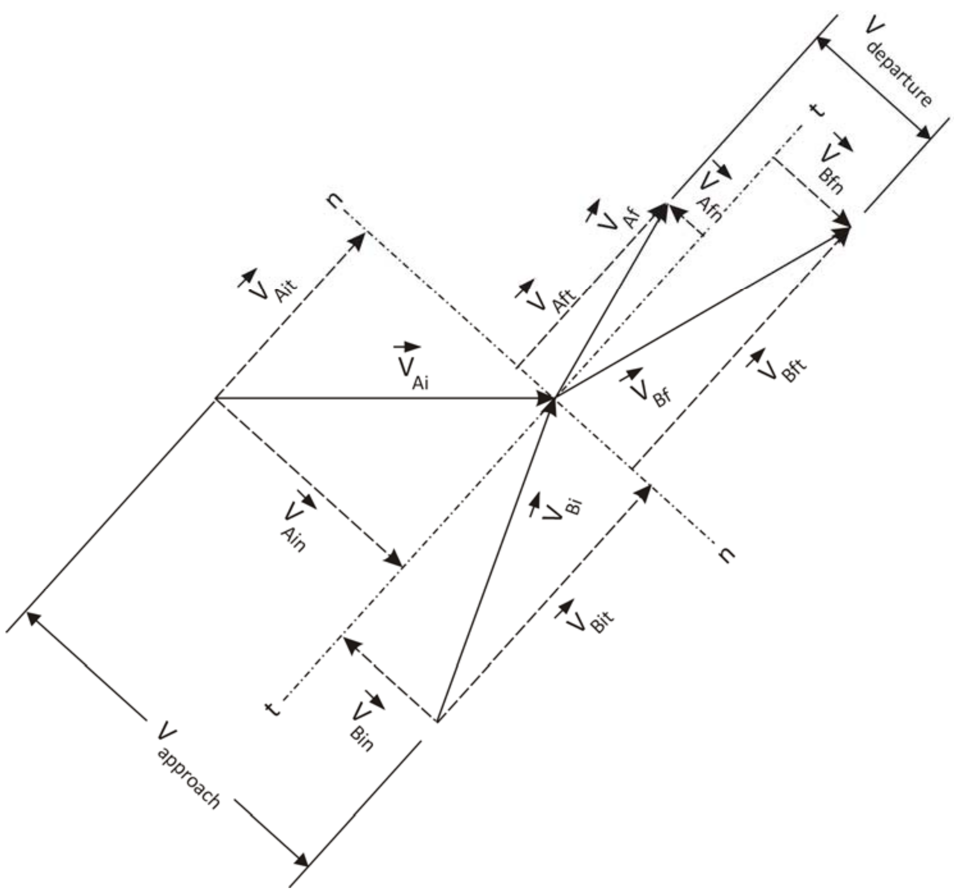


Figure 7 – Velocity diagram

in constructing this figure because the momentum vectors for each vehicle are scaled differently by the mass of each. If you compare the above figure with the momentum diagram from which it was extracted, you will see that the A-vectors, when compared with the B-vectors, are larger on the velocity diagram than they are on the momentum diagram. This means that vehicle A is smaller than vehicle B. Since $V = p/m$, the smaller mass in the denominator magnifies the velocity vectors for vehicle A.

Figure 7 shows two directions, *n*, or *normal*, which is perpendicular to the collision plane, and *t*, or *tangential*, which is tangential to the collision plane. The *n*-direction is the direction of the PDOF. The initial and final velocities of each vehicle are broken up into *n* and *t* components. The velocities of approach and departure are scalar quantities for the purpose of calculating the coefficient of restitution of the crash. An often-cited figure for this is 0.1. If a value of *e* is calculated that is quite different, one must ask why. *e* is a measure of the spring-back nature of the vehicles after maximum crushing during the crash.

Another interesting feature of the crash that is made clear by Figure 7 is the following. Since the force (and thus impulse) between the vehicles is in the *n*-direction, there is no force or impulse in the *t*-direction. With no impulse in the *t*-direction, there is no change in linear momentum in the *t*-direction. Thus there is no change in each vehicle's velocity component in the *t*-direction during the crash. So

$$\vec{V}_{Aft} = \vec{V}_{Ait}$$

and

$$\vec{V}_{Bft} = \vec{V}_{Bit}$$

This can be verified by checking the figure. This is also a useful check to make of the figure. If these vectors appear not to be the same length, one should search for a mistake in the construction of the figure.

Conclusion

In accident reconstruction nowadays the momentum calculation is often done with computer software. But this diagram is still useful for the analyst to check the computer/calculator solution. It gives the analyst also a graphical feel for a collision's dynamics that can be missing nowadays, when reliance on computers and calculators is often too great. The momentum diagram is still useful as a graphical reminder of how the vectors must fit together so as not to violate conservation of linear momentum in a collision.

As has been shown here, there is a great deal of useful information that can be extracted from a momentum diagram. There are a number of checks that can be using the diagram that safeguard against erroneous results.

Example

This example is taken from "Test Your Skill" in the September/October, 2011 Accident Reconstruction Journal. An eastbound vehicle, V-A, is struck broadside by a southbound vehicle, V-B. The vehicle particulars are given in the following table:

	V-A	V-B
$W_{B\text{-}loaded}$ (weight, lb)	4140	2990
ψ_i (approach angle)	90°	180°
ψ_f (departure angle)	143°	156°
d_f (post-collision skid distance, ft)	38	42
f (drag factor)	0.48	0.40

The heading angles are measured clockwise from north.

Solution

Start by finding the post-collision velocities of each vehicle. The departure angles are given, having been determined from skid marks. An energy balance using the drag force and the post-collision kinetic energy gives the vehicle speeds just after the collision.

$$\frac{1}{2} m \cdot V_f^2 = f \cdot W \cdot d_f$$

$$V_f = \sqrt{2 \cdot f \cdot g \cdot d_f}$$

This results in

$$V_{Af} = 34.3 \text{ fps} = 23.4 \text{ mph} \quad \text{and} \quad V_{Bf} = 32.9 \text{ fps} = 22.4 \text{ mph}$$

The angles for these two vectors are the departure angles in the table. The linear momentum of each vehicle is then calculated by multiplying the velocity vectors by the vehicle masses, respectively. Thus

$$\vec{p}_{Af} = \frac{W_A}{g} \vec{V}_{Af} = 4407 \text{ lb·sec @ } 143°$$

$$\vec{p}_{Bf} = \frac{W_B}{g} \vec{V}_{Bf} = 3054 \text{ lb·sec @ } 156°$$

These two vectors are drawn to start the momentum diagram (step 1, shown in the figure below). They are then summed to get the post-collision linear momentum of both cars together.

$$\vec{p}_{Af} + \vec{p}_{Bf} = \vec{p}_f = 7414 \text{ lb·sec @ } 148°$$

This is also drawn on the diagram (step 2). Since linear momentum of the entire system is conserved, the pre-collision momentum is drawn simply by copying the post-collision momentum vector (step 3).

Since the pre-collision velocity directions of the two vehicles are known, and since their sum must be equal to $\vec{p}_i$, the individual pre-impact momentum vectors can be constructed (step 4). So

$$\vec{p}_{Ai} = p_i cos32° \text{ @ } 90° = 3894 \text{ lb·sec @ } 90°$$

$$\vec{p}_{Bi} = p_i sin32° \text{ @ } 180° = 6309 \text{ lb·sec @ } 180°$$

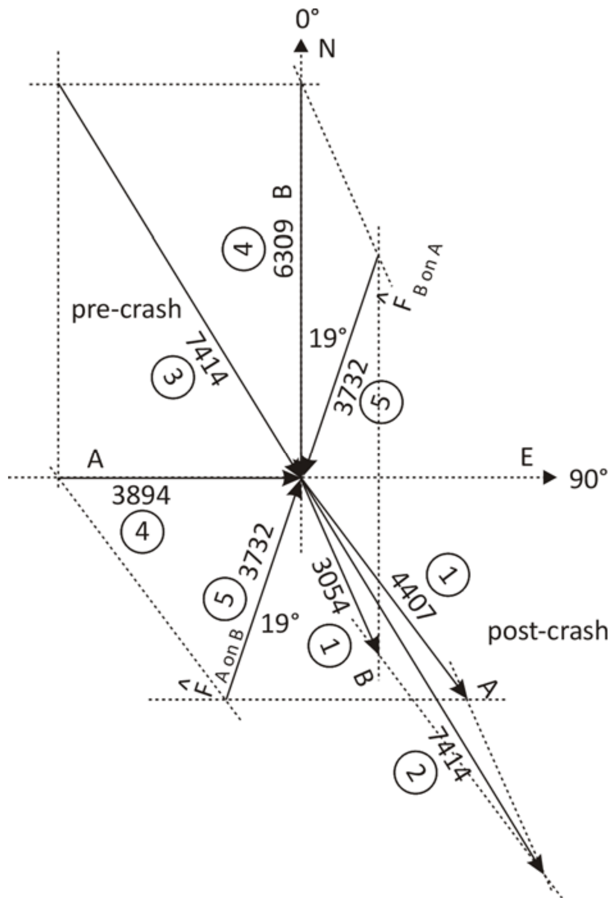


Figure 8 – Momentum diagram for example

Now the focus turns to each vehicle and the impulse force that caused its change in momentum. Since

$$\vec{p}_i + \hat{F} = \vec{p}_f$$

$$\hat{F} = \vec{p}_f - \vec{p}_i$$

Thus $\hat{F}$ is the vector sum of the final linear momentum and the initial momentum turned 180° from the direction shown from step 4 (because of $\vec{p}_i$'s minus sign in the equation above). These can be constructed on the diagram as shown in step 5. The vector sum for vehicle A is

$$\hat{F}_A = 4407 \text{ lb·sec @ } 143° - 3894 \text{ lb·sec @ } 90° = 3732 \text{ lb·sec @ } 199°$$

This is the impact force (technically called the *impulse*) imposed on vehicle A by vehicle B. The angle of this force is thus the PDOF of vehicle A's impact force. $\hat{F}_B$ has the same magnitude but is oriented 180° from $\hat{F}_B$. Thus

$\hat{F}_B = 3054$ lb·sec @ 156° - 6309 lb·sec @ 180° = 3732 lb·sec @ 19°

Note that these two impact forces are equal in magnitude and opposite in direction.

Figure 9 shows the velocity diagram for this crash. Recall that this diagram is constructed by converting the linear momenta for each vehicle back to velocities by dividing the momenta by each vehicle's mass. This has been done for this case with the velocities also being converted into mph units. What can be observed by carefully comparing the velocity diagram with the momentum diagram is that the scaling for the A velocities is different from the scaling for the B velocities, owing to the different vehicle masses.

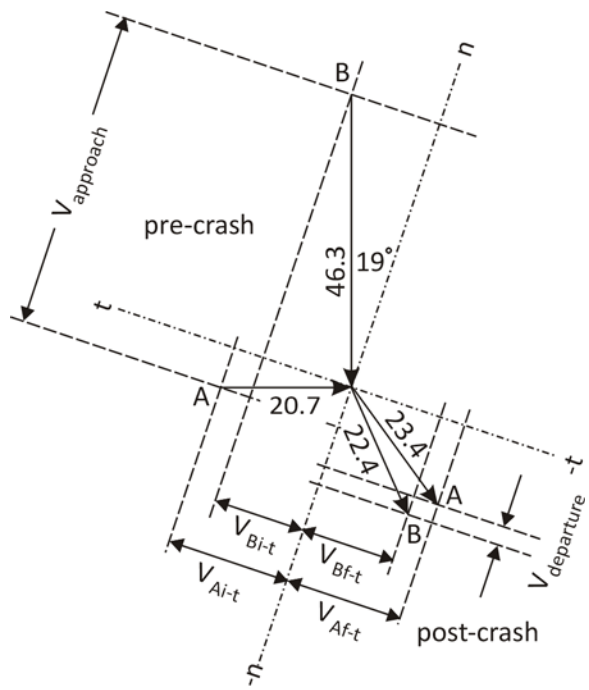


Figure 9 – Velocity diagram for example

Note also that the physical plane of contact and the direction normal to this have been constructed, with the *n*-direction defined by the PDOF and the *t*-direction perpendicular to this. Thus the two equal-and-opposite impulses act along the *n*-axis. This means that there is no force in the *t*-direction. Since there is no force in this direction, there is no change in momentum in the *t*-direction also. So for each vehicle $V_{f\text{-}t} = V_{i\text{-}t}$. This is shown clearly in the lower central part of Figure 9.

$$e = \frac{V_{departure}}{V_{approach}} = \frac{V_{Bf-n} - V_{Af-n}}{V_{Ai-n} - V_{Bi-n}} = \frac{-16.3 - (-12.9)}{6.9 - (-43.7)} = -0.06657$$

If we look in the *t*-direction, we can calculate *e*, the coefficient of restitution, in the crash. As the above result shows, *e* is calculated to be negative. This is somewhat unusual, since the cars have to actually move toward each other in the normal direction after the crash. But they indeed do this, as is clear in the diagram of the problem that appears in the Accident Reconstruction Journal.

A modification of the velocity diagram shows the determination of ΔV. Figure 10 shows the pre- and post-crash velocities. As shown and expected, the ΔVs are along the *n*-axis, since they are just the impulses of each vehicle scaled by the inverse of each mass, respectively.

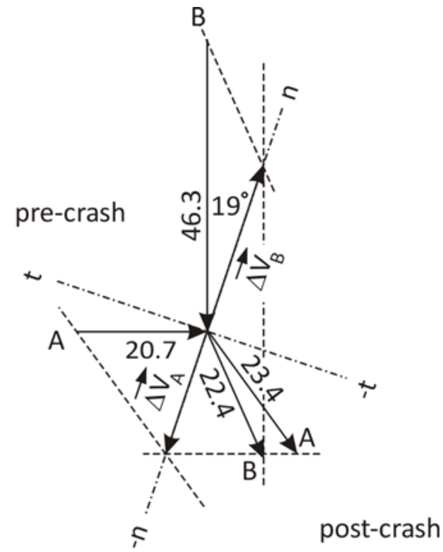


Figure 10 – Determination of ΔVs from velocity diagram

Thus the magnitudes are

$$\Delta V_A = \sqrt{(20.7 + 23.4\cos(53))^2 + (23.4\sin(53))^2} = 19.8\ mph$$

$$\Delta V_B = \sqrt{(22.4\sin(66))^2 + (46.3 + 22.4\cos(66))^2} = 27.4\ mph$$

These can be checked by dividing the impulses by the respective masses of the vehicles.

One last word regarding the *n*- and *t*-directions. One should inspect the vehicles involved in the collision to verify that there is some rough correspondence of the *n*-direction as the PDOF, perhaps by seeing if something like a flat face of contact is evident in the damage to each vehicle.

References

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